

Spikes in Cosmic Crystallography

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Abstract

If the universe is multiply connected and small the sky shows multiple images of cosmic objects, correlated by the covering group of the 3-manifold used to model it. These correlations were originally thought to manifest as spikes in pair separation histograms (PSH) built from suitable catalogues. Using probability theory we derive an expression for the expected pair separation histogram (EPSH) in a rather general topological-geometrical-observational setting. As a major consequence we show that the spikes of topological origin in PSH's are due to translations, whereas other isometries manifest as tiny deformations of the PSH corresponding to the simply connected case. This result holds for all Robertson-Walker spacetimes and gives rise to two basic corollaries: (i) that PSH's of Euclidean manifolds that have the same translations in their covering groups exhibit identical spike spectra of topological origin, making clear that even if the universe is flat the topological spikes alone are not sufficient for determining its topology; and (ii) that PSH's of hyperbolic 3-manifolds exhibit no spikes of topological origin. These corollaries ensure that cosmic crystallography, as originally formulated, is not a conclusive method for unveiling the shape of the universe. We also present a method that reduces the statistical fluctuations in PSH's built from simulated catalogues.

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1 Introduction

Current observational data favour Friedmann-Lemaître (FL) cosmological models as approximate descriptions of our universe at least since the recombination time. These descriptions are, however, only local and do not fix the global shape of our universe. Despite the infinitely many possibilities for its global topology, it is often assumed that spacetime is simply connected leaving aside the hypothesis, very rich in observational and physical consequences, that the universe may be multiply connected, and compact even if it has zero or negative constant curvature. Since the hypothesis that our universe has a non-trivial topology has not been excluded, it is worthwhile testing it (see [1] – [3] and references therein).

The most immediate consequence of the hypothesis of multiply-connectedness of our universe is the possibility of observing multiple images of cosmic objects, such as galaxies, quasars, and the like. Thus, for example, consider the available catalogues of quasars with redshifts ranging up to $z \approx 4$ which, in the Einstein-de Sitter cosmological model, corresponds to a comoving distance $d \approx 3300 h^{-1} Mpc$ from us (h is the Hubble constant in units of $100 km s^{-1} Mpc^{-1}$). Then, roughly speaking, if our universe is *small* in the sense that it has closed geodesics of length less than $2d$, some of the observed quasars may actually be images of the same cosmic object.¹

More generally, in considering discrete astrophysical sources, the *observable universe* can be viewed as that part of the universal covering manifold $\widetilde{M}(t_0)$ of the $t = t_0$ space-like section $M(t_0)$ of spacetime, causally connected to an image of our position since the moment of matter-radiation decoupling (here t_0 denotes present time) while, given a catalogue of cosmic sources, the *observed universe* is that part of the observable universe which contains all the sources listed in the catalogue. So, for instance, using quasars as cosmic sources the observed universe for a catalogue covering the entire sky is a ball with radius approximately half the radius of the observable universe (in the Einstein-de Sitter model). If the universe $M(t_0)$ is small enough in the above-specified sense, then there may be *copies* of some cosmic objects in the observed universe, and an important goal in observational cosmic topology is to develop methods to determine whether these copies exist.

Direct searching for multiple images of cosmic objects is not a simple problem. Indeed, due to the finiteness of the speed of light two images of a given object at different distances correspond to different epochs of its life. Moreover, in general the two images are seen from different directions. So one ought to be able to find out whether two images correspond

¹Note that we are not considering problems which arise from the possibly short lifetime of quasars. Actually, this is irrelevant for the point we want to illustrate with this example.

to different objects, or correspond to the same object seen at two different stages of its evolution and at two different orientations. The problem becomes even more involved when one takes into account that observational and selection effects may also be different for these distinct images.

One way to handle these difficulties is to use suitable statistical analysis applied to catalogues. Cosmic crystallography [4] is a promising statistical method which looks for distance correlations between cosmic sources using pair separation histograms (PSH), i.e. graphs of the number of pairs of sources versus the squared distance between them. These correlations are expected to arise from the isometries of the covering group of $M(t_0)$ which give rise to the (observed) multiple images, and have been claimed to manifest as sharp peaks [4], also called spikes. Moreover, the positions and relative amplitudes of these spikes have also been thought to be fingerprints of the shape of the universe (see, however, the references [5] – [6], and also sections 5 and 6 of the present paper).

It should be emphasized that just by examining a single PSH one cannot at all decide whether a particular spike is of topological origin or simply arises from statistical fluctuations. Actually, depending on the accuracy of the simulation one can obtain PSH's with hundreds of sharp peaks of purely statistical origin among just a few spikes of topological nature. Thus, it is of indisputable importance to perform a theoretical statistical analysis of the distance correlations in the PSH's at least to have a criterion for revealing the ultimate origin of the spikes that arise in the PSH's and pave the way for further refinements of the crystallographic method.

In this work, by using the probability theory we derive the expression (4.13) for the expected pair separation histogram (EPSH) of comparable catalogues with the same number of sources and corresponding to any complete 3-manifold of constant curvature. The EPSH, which is essentially a PSH from which the statistical noise has been withdrawn, is derived in a very general topological-geometrical-observational setting. It turns out that the EPSH built from a multiply connected manifold is an EPSH in which the contributions arising from the correlated images were withdrawn, plus a term that consists of a sum of individual contributions from each covering isometry.

From the EPSH (4.13) we extract its major consequence, namely that the sharp peaks (or spikes) of topological nature in individual pair separation histograms are due to Clifford translations, whereas all other isometries manifest as tiny deformations of the PSH of the corresponding universal covering manifold. This relevant consequence holds for all Robertson-Walker (RW) spacetimes and in turn gives rise to two others: (i) that Euclidean distinct manifolds which have the same translations in their covering groups present the same spike spectra of topological nature. So, the set of topological spikes (their positions

and relative amplitudes) in the PSH alone is not sufficient for distinguishing these compact flat manifolds, making clear that even if the universe is flat ($\Omega_{tot} = 1$) the spike spectrum is not enough for determining its global shape; and (ii) that individual PSH's corresponding to hyperbolic 3-manifolds exhibit no spikes of topological origin, since there are no Clifford translations in the hyperbolic geometry. These two corollaries ensure that cosmic crystallography, as originally formulated, is not a conclusive method for unveiling the shape of the universe.

As a way to reduce the statistical fluctuations in individual PSH's so as to unveil the contributions of non-translational isometries to the topological signature, we introduce the mean pair separation histogram (MPSH) and show that it is a suitable approximation for the EPSH. Moreover, we emphasize that the use of MPSH's is restricted to simulated catalogues due to the unsurmountable practical difficulties in constructing several comparable catalogues of real sources.

The lack of spikes of topological origin in PSH's of multiply connected hyperbolic 3-manifolds has also been found in histograms for the specific cases of Weeks [5] and one of the Best [6] manifolds, making apparent that, within the degree of accuracy of the corresponding plots, the above corollary (ii) holds for these specific cases. Concerning these references, we also discuss the connection between ours and their [5, 6] results. Further, we point out the limitation of the set of conclusions one can withdraw from such graphs by using, e.g., the specific PSH shown in fig. 1 of [7]. The possible origins for the spikes in PSH's are also discussed, and it is shown that one can distinguish between statistical sharp peaks (noise) and topological spikes only through the use of the rather general results obtained in this article.

The plan of this paper is as follows. In the next section we describe what a catalogue of cosmic sources is in the context of Robertson-Walker (RW) spacetimes, and introduce some relevant definitions to set our framework and make our paper as accurate and self-contained as possible. In the third section we describe how to construct a PSH from a given catalogue (either real or simulated), and discuss qualitatively how distance correlations arise in multiply connected RW universes. In the fourth section we first discuss the concept of *expected* pair separation histogram (EPSH) and derive its explicit expression in a very general topological-geometrical-observational setting. In the fifth section we use the expression for the EPSH obtained in section 4 to derive its most general consequence, namely that spikes of topological origin in PSH's are due to *translations* alone. We proceed by analyzing how this general result affects PSH's built from Euclidean and hyperbolic manifolds, and derive two relevant consequences regarding these classes of manifolds. We also present in that section the MPSH as a simple approach aiming at reducing the statistical

fluctuations in PSH's so that the contributions from non-translational isometries become apparent. In section 6 we summarize our conclusions, briefly indicate possible approaches for further investigations, and finally discuss the connection between ours and the results those reported in [5] and [6].

2 Catalogues in multiply connected RW spacetimes

If the universe is multiply connected and one can form catalogues of cosmic sources with multiple images, the problem of identifying its shape can be reduced to that of designing suitable methods for extracting the underlying topological information from these catalogues. In this section we describe what a catalogue of cosmic sources is in the context of RW spacetimes, and discuss under what conditions catalogues present multiple images. We first briefly review some basic properties of locally homogeneous and isotropic cosmological models, then we formalise the practical process of construction of catalogues of discrete astrophysical sources, and finally we specify the conditions for the existence of multiple images in a catalogue.

Locally homogeneous and isotropic cosmological models

The spacetime arena for a FL cosmological model is a 4-dimensional manifold endowed with a RW metric which can be written locally as

$$ds^2 = dt^2 - a^2(t) d\sigma^2, \quad (2.1)$$

where t is a cosmic time, $d\sigma$ is a standard 3-dimensional hyperbolic, Euclidean or spherical metric, and $a(t)$ is the scale factor that carries the unit of length. It is usually assumed that the $t = \text{const}$ spatial sections of a RW spacetime are one of the following simply connected spaces: hyperbolic (H^3), Euclidean (E^3), or the 3-sphere (S^3), depending on the local curvature computed from $d\sigma$. Cosmic topology arises when we relax the hypothesis of simply-connectedness and consider that the $t = \text{const}$ spatial sections may also be any complete multiply connected 3-manifold of constant curvature (see, for example, [8, 9]).

In this work we shall consider spacetimes of the form $I \times M$, with I a (possibly infinite) open interval of the real line, and M a complete constant curvature 3-manifold, either simply or multiply connected. Actually, a RW spacetime is a warped product $I \times_a M$ (see, e.g., [10] for more details) in that, for any instant $t \in I$, the metric in M is $d\sigma(t) = a(t)d\sigma$. The manifold M equipped with the metric $d\sigma(t)$ is denoted by $M(t)$, and is called *comoving space* at time t . So, *comoving geodesics* and *comoving distances* at some time t mean, respectively, geodesics of $M(t)$ and distances between points on $M(t)$. Throughout this

article we will omit the time dependence of $M(t)$ and $\widetilde{M}(t)$ whenever these manifolds are endowed with the standard metric $d\sigma$.

Before proceeding to the discussion of the notion of catalogues we recall that in a FL cosmological model the energy-matter content and Einstein's field equations determine the local curvature of the spatial sections and the scale factor $a(t)$. Thus in this process of cosmological modelling within the framework of general relativity the introduction of particular values for cosmological parameters (H_0 , Ω_m , Ω_Λ , q_0 , and so forth) restricts both the locally homogeneous-and-isotropic 3-geometry, and the scalar factor $a(t)$. However, the concepts and results we shall introduce and derive in this work hold regardless of the particular 3-geometry and of the form of $a(t)$ provided that $a(t)$ is a monotonically increasing function, at least since the recombination time. So, there is definitely no need to introduce any particular values for the cosmological parameters unless one intends to examine the consequences of a specific class of RW models, which for the sake of generality we do not aim at in the present article.

Constructing catalogues

To formalise the concept of catalogue of cosmic sources, let us assume that we know the scale factor $a(t)$, and that it is a monotonically increasing function. We shall also assume that all cosmic objects of our interest (also referred to simply as objects) are pointlike and have long lifetimes so that none was born or dead within the time interval given by I . Moreover, we shall also assume that all objects are comoving, so that their worldlines have constant spatial coordinates. Although unrealistic, these assumptions were used implicitly in [4] and [7] and are very useful to study the observational consequences of a non-trivial topology for the universe.

It should be noted that in the process of construction of catalogues we shall describe below it is assumed that a particular type of sources (clusters of galaxies, quasars, etc) or some combination of them (quasars and BL Lac objects, say) is chosen from the outset. This approach does not coincide with the exact manner the astronomers build catalogues, in that usually they simply record any sources within their range of interest and (or) observational limitations. However, the model of constructing catalogues we shall present relies on the fact that any catalogue of a specific type of sources ultimately is a selection of sources of that type from the hypothetical complete set of sources which can in principle be observed.

Since we are assuming that all objects are comoving their spatial coordinates are constant. So, the set of all the objects in M is given by a list of their present comoving

positions, and from this list one can define a map

$$\begin{aligned} \mu : M(t_0) &\rightarrow \{1, 0\} \\ p \mapsto \mu(p) &= \begin{cases} 1 & \text{if there is an object at } p, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (2.2)$$

The set of objects in $M(t_0)$ is thus $\mu^{-1}(1)$. This is a discrete set in $M(t_0)$ without accumulation points. Actually, from any map $\mu : M(t_0) \rightarrow \{1, 0\}$ such that $\mu^{-1}(1)$ is a discrete set without accumulation points, one may define a set of objects, namely the set $\mu^{-1}(1)$. We will further assume in this work that the set of objects is a representative sample of some well-behaved distribution law in $M(t_0)$. For our purposes in this article a distribution is well-behaved if it gives rise to samples of points which are not concentrated in small regions of $M(t_0)$.

Let $\pi : \widetilde{M}(t_0) \rightarrow M(t_0)$ be the universal covering projection of $M(t_0)$ and p be an object, that is $p \in \mu^{-1}(1)$. The set $\pi^{-1}(p)$ is the collection of *copies* of p on $\widetilde{M}(t_0)$. We will refer to these copies as *topological images* or simply as images of the object p , thus the map $\tilde{\mu}$ defined by the commutative diagram

$$\begin{array}{ccc} \widetilde{M}(t_0) & & \\ \pi \downarrow & \searrow \tilde{\mu} & \\ M(t_0) & \xrightarrow{\mu} & \{1, 0\} \end{array}$$

gives the set of all images on the universal covering manifold $\widetilde{M}(t_0)$.² Indeed, the images of the objects in $M(t_0)$ are the elements of the set $\tilde{\mu}^{-1}(1)$.

²It has occasionally been used in the literature a misleading terminology in which the topological images are classified as real and ghosts. In most cases the expression ‘real image’ refers to the nearest topological image of a given object, while the expression ‘ghost image’ refers to any other image of the same object. There are also cases where ‘real images’ has been used to refer to the topological images which lie inside a fundamental polyhedron (FP), while ‘ghost images’ refers to the images outside the FP. This latter classification of the images depends on the choice of the FP which is not at all unique. Actually, these two usages for the expressions ‘real image’ and ‘ghost images’ are compatible only if the FP is the Dirichlet domain with centre at an image of the observer [11]. In both cases the terminology is misleading also because it may suggest that either the nearest images or the images inside a FP are somehow special. And yet there is no physical or geometrical property which supports this distinction. Besides, this terminology is unnecessary since real objects are represented by points which lie on the manifold M , whereas the points on the universal covering \widetilde{M} can represent only (topological) images of these objects — no image is more *real* than the other, they are simply (topological) images.

Now suppose we perform a full sky coverage survey for the objects up to a redshift cutoff z_{max} . Since we are assuming that we know the metric $a(t)d\sigma$, we can compute the distance R corresponding to this redshift cutoff and so determine the observed universe corresponding to this survey. The ball $\mathcal{U} \subset \widetilde{M}(t_0)$ with radius R and centred at an image of our position is a representation of this observed universe. The finite set $\mathcal{O} = \tilde{\mu}^{-1}(1) \cap \mathcal{U}$ is the set of *observable images* since it contains all the images which can in principle be observed up to a distance R from one image of our position.

The set of *observed images* or *catalogue* is a subset $\mathcal{C} \subset \mathcal{O}$, since by several observational limitations one can hardly record all the images present in the observed universe. Our observational limitations can be formulated as *selection rules* which describe how the subset \mathcal{C} arises from \mathcal{O} . These selection rules, together with the distribution law which the objects in M obey, will be referred to as *construction rules* for the catalogue \mathcal{C} . A good example of construction rules appears in the simulated catalogue constructed in ref. [4] where an uniform distribution of points in a 3-torus is assumed together with a selection rule which dictates how one obtains a catalogue \mathcal{C} from the set of images in an observed universe \mathcal{U} subjected to (defined by) the redshift cutoff $z_{max} = 0.26$. In that example, to mimic the obscuration effect by the galactic plane, they have taken as selection rule that only the images inside a double cone of aperture 120° are observed or considered. However, in more involved simulations, one can certainly take other selection rules such as, e.g., luminosity threshold, finite lifetime and the obscuration by the line of sight, and (or) a combination of them.

Throughout this work we shall assume that catalogues obey well-defined construction rules, and we shall say that two catalogues are *comparable* when they are defined by the same construction rules; even if they have a significantly different number of sources and correspond to possibly different (topologically) 3-manifolds compatible with a given 3-geometry. According to this definition two comparable catalogues must correspond to a given RW geometry (2.1) with obviously a well-defined scale factor, and the same underlying redshift cutoff. In other words, comparable catalogues correspond to a precise set of cosmological parameters of an observed universe, plus a fixed redshift cutoff. The main motivation for formalising in this way the concept of comparable catalogues comes from the fact that in the cosmic crystallographic method we are often interested in comparing PSH's from simulated catalogues against PSH's from real catalogues. And real catalogues are limited by a redshift cutoff that is converted into distance through an *ad hoc* choice of a RW geometry. So, to build simulated catalogues comparable to a specific real catalogue, for example, one has to begin with the precise RW geometry that transforms redshifts into distances in the real catalogues, and use the same redshift cutoff of the underlying real

catalogue.

Finally, it should be noticed that the above definition for a catalogue fits in with the two basic types of catalogues one usually finds in practice, namely *real catalogues* (arising from observations) and *simulated catalogues*, which are generated under well-defined assumptions that are posed to mimic some observational limitations and (or) to account for simplifying hypotheses.

Catalogues with multiple images

If M is simply connected then M and \widetilde{M} are the same, and so there is exactly one image for each object. If M is multiply connected then each object has several images (actually an infinite number of images in the cases of zero and negative curvature). Suppose that M is multiply connected, and let $P \subset \widetilde{M}(t_0)$ be a fundamental domain of $M(t_0)$. P can always be chosen in such a way that $\tilde{\mu}^{-1}(1) \cap \partial P = \emptyset$, where ∂P is the boundary of P . If we consider the universal covering $\widetilde{M}(t_0)$ tessellated by P , then clearly $\tilde{\mu}^{-1}(1)$ presents all the periodicities due to the covering group Γ , in the sense that in each copy gP of the fundamental domain ($g \in \Gamma$) there is the same distribution of images as in P .

To be able to guarantee the existence of multiple images in a catalog \mathcal{C} , we shall need the concept of a *deep enough* survey, which is a survey whose corresponding observed universe \mathcal{U} has the property that for some fundamental polyhedron P , there are faces F and F' , identified by an isometry $g \in \Gamma$, and such that some portions $E \subset F$ and $g(E) \subset F'$ are in the interior of \mathcal{U} . In particular when M is compact with some fundamental polyhedron lying inside the observed universe \mathcal{U} , then this observed universe corresponds to a deep enough survey. To find out whether a full sky coverage survey is deep enough, in practice, all one needs to do is to determine the closest image of our position and using the metric $a(t)d\sigma$ compute the redshift z_{thr} corresponding to half of that distance. Any full sky coverage survey with redshift $z > z_{thr}$ is said to be a deep enough survey.³

When M is multiply connected and the survey is deep enough the set of observable topological images \mathcal{O} contains multiple images of some cosmic objects. If, in addition, our observational capabilities allow the presence of multiple images in \mathcal{C} , then the catalogue has information on the periodicities due to the covering group Γ , and so about the manifold M . Every pair of images of one object is related by an isometry of Γ . These pairs of images have been called *gg-pairs* [4], however when referring to them collectively we shall use the term Γ -pairs, reserving the name *g-pair* for any pair related by a specific isometry $g \in \Gamma$.

³As a matter of fact z_{thr} is the redshift corresponding to the radius of the inscribed ball in the Dirichlet polyhedron of M centred in an image of our position [11].

The Γ -pairs in \mathcal{C} give rise to correlations in the positions of the observed images. The main goal of any statistical approach to cosmic topology based on discrete sources is to develop methods to reveal these correlations. Cosmic crystallography is one such method, and uses PSH's to obtain the *distance correlations* which arise from these correlations in positions.

It should be stressed that there are two independent conditions that must be satisfied to have multiple images in a catalogue \mathcal{C} . Firstly, the survey has to be deep enough, so that in the observed universe \mathcal{U} there must exist multiple observable images of cosmic objects. Secondly, the selection rules, which dictate how one obtains a catalogue \mathcal{C} from the observed universe \mathcal{U} , must not be so restrictive as to rule out the possible multiple images \mathcal{C} . Clearly if the survey is not deep enough there is no chance of having multiple images in \mathcal{C} , regardless of the quality of the observations. On the other hand, even when the survey is deep enough, if the selection rules are too strict they may reduce the multiple images in \mathcal{C} to a level that the detection of topology becomes impossible.

Finally, it should be noticed that when M is simply connected, to any cosmic object corresponds just one image, so in this case there is an one-to-one correspondence between images and objects, and we can simply identify them as the same entity. Since we do not know a priori whether our universe is simply or multiply connected, we do not know if we are recording objects or just images in real catalogues, hence we say that a catalogue is formed by *cosmic sources*.

3 Pair separation histograms

The purpose of this section is two-fold. First we shall give a brief description of what a PSH is and how to construct it. Then we shall describe qualitatively how distance correlations arise in a particular multiply connected universe, motivating therefore the statistical analysis we will perform in the next section.

To build a PSH we simply evaluate a suitable one-to-one function f of the distance r between the cosmic sources of every pair from a given catalogue \mathcal{C} , and then count the number of pairs for which these values $f(r)$ lie within certain subintervals. These subintervals must form a partition of the interval $(0, f(2R)]$, where R is the distance from us to the most distant source in the catalogue. Usually all the subintervals are taken to be of equal length. The PSH is just a plot of this counting. Actually, what we shall call a PSH is a normalized version of this plot. The function f is usually taken to be the square function, whereas for very deep catalogues it might be convenient to try some hyperbolic function if we are, for example, dealing with open FLRW models. In line with the usage in the literature and to be specific, in what follows we will take f to be the square function.

However, it should be emphasized that the results we obtain here and in the next section hold regardless of this particular choice.

A formal description of the above procedure is as follows. Given a catalogue \mathcal{C} of cosmic sources we denote by $\eta(s)$ the number of pairs of sources whose squared separation is s . Formally, this is given by the function

$$\begin{aligned} \eta : (0, 4R^2] &\rightarrow [0, \infty) \\ s &\mapsto \frac{1}{2} \text{Card}(\Delta^{-1}(s)) , \end{aligned}$$

where, as usual, $\text{Card}(\Delta^{-1}(s))$ is the number of elements of the set $\Delta^{-1}(s)$, and Δ is the map

$$\begin{aligned} \Delta : \mathcal{C} \times \mathcal{C} &\rightarrow [0, 4R^2] \\ (p, q) &\mapsto d^2(p, q) . \end{aligned}$$

Clearly, the distance $d(p, q)$ between sources $p, q \in \mathcal{C}$ is calculated using the geometry one is concerned with. The factor $1/2$ in the definition of η accounts for the fact that the pairs (p, q) and (q, p) are indeed the same pair.

The next step is to divide the interval $(0, 4R^2]$ in m equal subintervals of length $\delta s = 4R^2/m$. Each subinterval has the form

$$J_i = (s_i - \frac{\delta s}{2}, s_i + \frac{\delta s}{2}] \quad ; \quad i = 1, 2, \dots, m ,$$

with centre

$$s_i = (i - \frac{1}{2}) \delta s .$$

The PSH is then obtained from

$$\Phi(s_i) = \frac{2}{N(N-1)} \frac{1}{\delta s} \sum_{s \in J_i} \eta(s) , \quad (3.1)$$

where N is the number of sources in the catalogue \mathcal{C} . The coefficient of the sum is a normalization constant such that

$$\sum_{i=1}^m \Phi(s_i) \delta s = 1 . \quad (3.2)$$

Note that the sum in (3.1) is just a counting of the number of pairs of sources separated by a distance whose square lies in the subinterval J_i , hence $\Phi(s_i)$ is a normalized counting.

It should be stressed that throughout this paper we use normalized histograms instead of just plots of countings as in [4] and [7]. In doing so we can compare histograms built up from catalogues with a different number of sources. Further, although the PSH is actually

the plot of the function $\Phi(s_i)$, the function $\Phi(s_i)$ itself can be looked upon as the PSH. So, in what follows we shall refer to $\Phi(s_i)$ simply as the PSH.

As mentioned in the previous section, in a multiply connected universe the periodic distribution of images on \widetilde{M} (due to the covering group) gives rise to correlations in their positions, and these correlations can be translated into correlations in distances between pairs of images. For a better understanding on how these distance correlations arise let us consider the same example used by Fagundes and Gausmann [7] to clarify the method of cosmic crystallography. In their work they have assumed that the distribution of objects in M is uniform and the catalogue is the whole set of observable images. In the Euclidean 3-manifold they have studied, take a g -pair (p, gp) such that for a generic point $p = (x, y, z)$ we have $gp = (x - L, -y + 2L, -z)$. The squared separation between these points is given by

$$d^2 = 5L^2 - 8Ly + 4y^2 + 4z^2. \quad (3.3)$$

From this equation we have the following: firstly, that the separation of any other neighboring g -pair⁴ (q, gq) will be close to d ; secondly, several distant g -pairs are separated by approximately the same distance; thirdly, one has that not all g -pairs are separated by the same distance (these items hold for the isometry g used in this example, of course). Actually, the separation of any g -pair in Euclidean geometry is independent of the pair only when the isometry g is a translation. One can sum up by stating that from this example one might expect that in general correlations associated to translations manifest as spikes in PSH's, whereas correlations due to other isometries will be evinced through small deviations from the histogram due to uncorrelated pairs. This conjecture will be proved in the following two sections.

The distribution of cosmic objects may not be exactly homogeneous, nor any catalogue will consist of all the observable sources. For instance, the objects may present some clustering or may obey a fractal distribution, while luminosity threshold and obscuration effects limit our observational capabilities. We shall show in the following section that the consideration of these aspects does not destroy the above-described picture, which was qualitatively inferred from general arguments and illustrated through the above specific example.

⁴We say that two g -pairs (p, gp) and (q, gq) are neighbors if the points p and q , and thus the points gp and gq , are neighbors.

4 The expected PSH

In this section we shall use elements from probability theory (see for example [12]) to show that the above qualitative description of the distance correlations in a PSH holds in a rather general framework. We shall make clear that this picture does not depend on the construction rules one uses to build a catalogue. Recall that the construction rules to build a catalogue \mathcal{C} from an observed universe \mathcal{U} consist of a well-behaved distribution law, of which the set of objects in $M(t_0)$ is a representative sample, and of selection rules which dictate how the catalogue is obtained from the set of all observable images \mathcal{O} .

The general underlying setting of the calculations in this section is the existence of an ensemble of catalogues comparable to a given catalogue \mathcal{C} (real or simulated), with the same number of sources N and corresponding to the same constant curvature 3-manifold $M(t_0)$. So, the construction rules permit the computation of probabilities and expected values of quantities which depend on the sources in the catalogue \mathcal{C} .

Our basic aim now is to compute the expected number, $\eta_{exp}(s_i)$, of observed pairs of cosmic sources in a catalogue \mathcal{C} of the ensemble with squared separations in J_i . Having $\eta_{exp}(s_i)$ we clearly have the *expected* pair separation histogram (EPSH) which is given by

$$\Phi_{exp}(s_i) = \frac{2}{N(N-1)} \frac{1}{\delta s} \eta_{exp}(s_i). \quad (4.1)$$

We remark that the EPSH carries all the relevant information of the distance correlations due to the covering group since

$$\Phi(s_i) = \Phi_{exp}(s_i) + \text{statistical fluctuations}, \quad (4.2)$$

where $\Phi(s_i)$ is the PSH constructed with \mathcal{C} .

It can be shown (see Appendix A for a proof) that the expected number $\eta_{exp}(s_i)$ can be decomposed into its uncorrelated part and its correlated part as

$$\eta_{exp}(s_i) = \eta_u(s_i) + \frac{1}{2} \sum_{g \in \tilde{\Gamma}} \eta_g(s_i), \quad (4.3)$$

where $\eta_u(s_i)$ is the expected number of observed uncorrelated pairs of sources with squared separations in J_i , i.e. pairs of sources that are not Γ -pairs; and $\eta_g(s_i)$ is the expected number of observed g -pairs whose squared separations are in J_i . $\tilde{\Gamma}$ is the covering group Γ without the identity map, and the factor $1/2$ in the sum accounts for the fact that, in considering all non-trivial covering isometries, we are counting each Γ -pair twice, since if (p, q) is a g -pair, then (q, p) is a (g^{-1}) -pair.

For each isometry $g \in \tilde{\Gamma}$ let us consider the function

$$\begin{aligned} X_g : \widetilde{M}(t_0) &\rightarrow [0, \infty) \\ p &\mapsto d^2(p, gp). \end{aligned}$$

This function is a random variable, and using the construction rules we can calculate the probability of an observed g -pair to be separated by a squared distance that lies in J_i ,

$$F_g(s_i) = P[X_g \in J_i] . \quad (4.4)$$

The construction rules allow us to compute also the expected number N_g of g -pairs in a catalogue \mathcal{C} with N sources. Clearly in a catalogue with twice the number of sources there will be $2N_g$ g -pairs. Actually, N_g is proportional to N so we write

$$N_g = N \nu_g , \quad (4.5)$$

with $0 \leq \nu_g < 1$. The expected number of observed g -pairs with squared separation in J_i is thus given by the product of N_g times the probability that an observed g -pair has its squared separation in J_i ,

$$\eta_g(s_i) = N \nu_g F_g(s_i) . \quad (4.6)$$

In order to examine uncorrelated pairs, we now consider the random variable

$$\begin{aligned} X : \widetilde{M}(t_0) \times \widetilde{M}(t_0) &\rightarrow [0, \infty) \\ (p, q) &\mapsto d^2(p, q) . \end{aligned}$$

The probability of an observed uncorrelated pair (p, q) to be separated by a squared distance that lies in J_i ,

$$F_u(s_i) = P[X \in J_i] , \quad (4.7)$$

can also be calculated from the construction rules. Thus, the expected number of observed uncorrelated pairs with squared separation in J_i is

$$\eta_u(s_i) = \left[\frac{1}{2} N(N-1) - \frac{1}{2} N \sum_{g \in \widetilde{\Gamma}} \nu_g \right] F_u(s_i) , \quad (4.8)$$

where we have used (4.6) and that clearly the expected number N_u of uncorrelated pairs is such that

$$N_u + \frac{1}{2} \sum_{g \in \widetilde{\Gamma}} N_g = \frac{1}{2} N(N-1) . \quad (4.9)$$

From (4.1),(4.3), (4.8) and (4.9) an explicit expression for the EPSH is thus

$$\Phi_{exp}(s_i) = \frac{1}{\delta_S} \frac{1}{N-1} \left[2 \frac{N_u}{N} F_u(s_i) + \sum_{g \in \widetilde{\Gamma}} \nu_g F_g(s_i) \right] , \quad (4.10)$$

where the sum in (4.10) is a finite sum since ν_g is nonzero only for a finite number of isometries. For multiply connected manifold M defining

$$\Phi_{exp}^u(s_i) = \frac{1}{\delta_S} F_u(s_i) , \quad (4.11)$$

$$\Phi_{exp}^g(s_i) = \frac{1}{\delta_S} F_g(s_i) , \quad (4.12)$$

we obtain a more descriptive expression for the EPSH, namely

$$\Phi_{exp}(s_i) = \frac{1}{N-1} [\nu_u \Phi_{exp}^u(s_i) + \sum_{g \in \tilde{\Gamma}} \nu_g \Phi_{exp}^g(s_i)] , \quad (4.13)$$

where by analogy with (4.5) we have defined $\nu_u = 2N_u/N$. From (4.13) it is apparent that the EPSH corresponding to a multiply connected manifold is an EPSH in which the contributions arising from the Γ -pairs have been withdrawn, plus a term that consists of a sum of individual contributions from each covering isometry.

When the manifold $M(t_0)$ which gives rise to the comparable catalogues is simply connected ($\tilde{\Gamma} = \emptyset$) all $N(N-1)/2$ pairs are uncorrelated, and so from (4.1) one clearly has

$$\Phi_{exp}^{sc}(s_i) = \frac{2}{N(N-1)} \frac{1}{\delta s} \eta_{exp}^{sc}(s_i) = \frac{1}{\delta s} F_{sc}(s_i) , \quad (4.14)$$

where $F_{sc}(s_i)$ is the probability that the two sources be separated by a squared distance that lies in J_i .

It turns out that the combination (which can be obtained from (4.9) and (4.13))

$$(N-1)[\Phi_{exp}(s_i) - \Phi_{exp}^{sc}(s_i)] = \nu_u [\Phi_{exp}^u(s_i) - \Phi_{exp}^{sc}(s_i)] + \sum_{g \in \tilde{\Gamma}} \nu_g [\Phi_{exp}^g(s_i) - \Phi_{exp}^{sc}(s_i)] \quad (4.15)$$

proves to exhibit a clear signal of the topology of $M(t_0)$ (see [15] – [17], which is already available in gr-qc archive, for more details, including numeric simulation and plots). This motivates the definition of the following quantity:

$$\varphi^S(s_i) = \nu_u [\Phi_{exp}^u(s_i) - \Phi_{exp}^{sc}(s_i)] + \sum_{g \in \tilde{\Gamma}} \nu_g [\Phi_{exp}^g(s_i) - \Phi_{exp}^{sc}(s_i)] , \quad (4.16)$$

which throughout this paper is referred to as *topological signature* of the multiply connected manifold $M(t_0)$, and clearly arises from the ensemble of catalogues of discrete sources \mathcal{C} . Equation (4.16) on the one hand makes explicit that (i) Γ -pairs as well as uncorrelated pairs which arise (both) from the covering isometries give rise to the topological signature; on the other hand, it ensures that (ii) the topological signature ought to arise in PSH's even when there are only a few images for each object.

5 Further results

From eqs. (4.4) and (4.12) we note that when g is a Clifford translation (i.e. an isometry such that for all $p \in \tilde{M}(t_0)$, the distance $|g| = d(p, gp)$ is independent of p) we have

$$\Phi_{exp}^g(s_i) = \begin{cases} 0 & \text{if } |g|^2 \notin J_i \\ (\delta s)^{-1} & \text{if } |g|^2 \in J_i . \end{cases} \quad (5.1)$$

Thus from equation (4.13) one has that the contribution of each translation g to the EPSH is a spike of amplitude $\nu_g [(N-1)\delta s]^{-1}$ at a well-defined subinterval J_{i_g} (say), minus a term proportional to the EPSH $\Phi_{exp}^u(s_i)$ for all $i = 1, \dots, i_g, \dots, m$. On the other hand, when g is not a Clifford translation, the separation $|g|$ depends smoothly on the g -pair because it is a composition of two smooth functions: the distance function and the isometry g . Moreover, the value it takes ranges over a fairly wide interval, so $F_g(s_i)$ will be non-zero for several subintervals J_i . In brief, from (4.13) we conclude that topological spikes in PSH's are due *only* to Clifford translations, whereas other isometries manifest as tiny deformations of the PSH corresponding to the simply connected case.

We emphasize that the above general result holds regardless of the underlying geometry, and for any set of construction rules. Further, when one restricts the above result to specific geometries, then one arrives at two rather important consequences, which we shall discuss in what follows.

Let us first consider the consequence for the particular case of Euclidean manifolds. It is known that any compact Euclidean 3-manifold M is finitely covered by a 3-torus [8]. Let Γ be the universal covering group of M , then the universal covering group of that 3-torus (consisting exclusively of translations) is a subgroup of Γ . Moreover, there is a covering 3-torus of M such that its covering group consists of all the translations contained in Γ .⁵ Thus, PSH's of M and of this minimal covering 3-torus, built from comparable catalogues with the same number of sources, have the same spike spectrum of topological origin, i.e. the same set of topological spikes with equal positions and amplitudes. Therefore the topological spikes alone are not sufficient for distinguishing these compact flat manifolds, making clear that even if the universe is flat ($\Omega_{tot} = 1$) the spike spectrum is not enough for determining its global shape.

Consider now the consequence of our major result for the special case of hyperbolic manifolds. Since there are no Clifford translations in hyperbolic geometry [13], there are no topological spikes in PSH's built from these manifolds. This result was not expected from the outset in that it has been claimed that spikes are a characteristic signature of topology in cosmic crystallography, at least for flat manifolds [4, 7] (see, however, in this regard the references [5] – [6]). As a matter of fact, this result is in agreement with simulations performed in the cases of Weeks [5] and Best [6] hyperbolic manifolds (see, however, next section for a comparison of ours and their results).

⁵The simplest example of this is a cubic *half-twisted 3-torus* [14], or cubic \mathcal{G}_2 manifold in Wolf's notation [8], which is double covered by a rectangular 3-torus with the same square base but with a height that is the double of the base side. This is easily seen by stacking two cubes defining \mathcal{G}_2 one on top of the other, the common face being the one that is twisted for an identification.

Incidentally, it can also be figured out from (4.13) that when g is not a translation, and since the probabilities $F_g(s_i)$ are non-zero for several subintervals J_i , then the contribution of these isometries to the EPSH are in practice negligible for $N \approx 2000$ as used in [4] and very small for $N \approx 250$ as used in [7]. In both papers these contributions are hidden by statistical fluctuations and, thus, are not revealed by the isolated PSH's they have plotted.

From what we have seen hitherto it turns out that cosmic crystallography, as originally formulated, is not a conclusive method for unveiling the shape of our universe since (i) in the Euclidean case the topological spikes will tell us that spacetime is multiply connected at some scale, leaving in some instances its shape undetermined, and (ii) in the hyperbolic case, as there are no translations (and therefore no topological spikes), it is even impossible to distinguish any hyperbolic manifold with non-trivial topology from the simply connected manifold H^3 . Improvements of the cosmic crystallography method are therefore necessary.

In the remainder of this section we will briefly discuss a first approach which refines upon that method. We look for a means of reducing the statistical fluctuations well enough to leave a visible signal of the non-translational isometries in a PSH. On theoretical ground, the simplest way to accomplish this is to use several comparable catalogues, with approximately equal number of cosmic sources, for the construction of a *mean* pair separation histogram (MPSH). For suppose we have K catalogues \mathcal{C}_k ($k = 1, 2, \dots, K$) with PSH's given by

$$\Phi_k(s_i) = \frac{2}{N_k(N_k - 1)} \frac{1}{\delta s} \sum_{s \in J_i} \eta_k(s) \quad (5.2)$$

with $N_k = \text{Card}(\mathcal{C}_k)$. The MPSH defined by

$$\langle \Phi(s_i) \rangle = \frac{1}{K} \sum_{k=1}^K \Phi_k(s_i) \quad (5.3)$$

is, in the limit $K \rightarrow \infty$, approximately equal to the EPSH. Actually, equality holds when all catalogues have exactly the same number of sources. However, it can be shown (see Appendix B for details) that even when the catalogues do not have exactly the same number N of sources, in first order approximation we still have

$$\Phi_{exp}(s_i) = \lim_{K \rightarrow \infty} \langle \Phi(s_i) \rangle . \quad (5.4)$$

Elementary statistics tells us that the statistical fluctuations in the MPSH are reduced by a factor of $1/\sqrt{K}$, which makes at first sight the MPSH method very attractive. It should be noticed, however, that since in practice it is not at all that easy to obtain many comparable real catalogues of cosmic sources, this method will hardly be useful in analyses which rely on real catalogues. On the other hand, since there is no problem in generating

hundreds of comparable (simulated) catalogues in a computer, the construction of MPSH's can easily be implemented in simulations, and so the MPSH method is a suitable approach for studying the role of non-translational isometries in PSH's. The use of the MPSH technique to extract the topological signature of non-translational isometries (including numeric simulation) is described in [15] – [17].

6 Conclusions and further remarks

In this section we begin by summarizing our main results, proceed by briefly indicating possible approaches for further investigations, and end by discussing the connection between ours and the results recently reported in the references [5] and [6].

Main results

In this work we have derived the expression (4.13) for the expected pair separation histogram (EPSH) for an ensemble of comparable catalogues with the same number of sources, and corresponding to spacetimes whose spacelike sections are any one of the possible 3-manifolds of constant curvature. The EPSH is essentially a typical PSH from which the statistical noise has been withdrawn, so it carries all the relevant information of the distance correlations due to the covering group of M . The EPSH (4.13) we have obtained holds in a rather general topological-geometrical-observational setting, that is to say it holds when the catalogues of the ensemble obey a well-behaved distribution law (needed to ensure that the sources are not concentrated in small regions) plus a set of selection rules (which dictate how the catalogues \mathcal{C} are obtained from the set of observable images \mathcal{O}). It turns out that the EPSH of a multiply connected manifold is an EPSH in which the contributions that arise from the Γ -pairs have been withdrawn, plus a sum of individual contributions from each covering isometry. From (4.13) and (4.16) we have also found that the topological signature (contribution of the multiply-connectedness) ought to arise in even when there are just a few images of each object.

Our theoretical study of distance correlations in pair separation histograms elucidates the ultimate nature of the spikes and the role played by isometries in PSH's. Indeed, from the expression (4.13) of the EPSH we obtain our major consequence, namely that the spikes of topological origin in single PSH's are only due to the translations of the covering group, whereas correlations due to the other (non-translational) isometries manifest as small deformations of the PSH of the underlying universal covering manifold. This result holds regardless of the (well-behaved) distribution of objects in the universe, and of the observational limitations that constrain, for example, the deepness and completeness of

the catalogues, as long as they contain enough Γ -pairs to yield a clear signal.

Besides clarifying the ultimate origin of spikes and revealing the role of non-translational isometries, the above-mentioned major result gives rise to two others:

- That Euclidean distinct manifolds which admit the same translations on their covering group present the same spike spectrum of topological nature. So, the set of topological spikes in the PSH's is not sufficient for distinguishing these compact flat manifolds, making clear that even if the universe is flat ($\Omega_{tot} = 1$) the spike spectrum may not be enough for determining its global shape;
- That individual pair separation histograms corresponding to hyperbolic 3-manifolds exhibit no spikes of topological origin, since there are no Clifford translations in hyperbolic geometry.

These two corollaries in turn ensure that cosmic crystallography, as originally formulated, is not a conclusive method for unveiling the shape of the universe and improvements of the method are thus necessary.

Any means of reducing the statistical noise well enough for revealing the correlations due to non-translational isometries should be in principle considered. Perhaps the simplest way to accomplish this is through the use of the MPSH method that we have also presented; that is to say by using several comparable catalogues to construct MPSH's. The major drawback of this approach, in practice, is the difficulty of constructing comparable catalogues of real sources. Nevertheless, the MPSH method is suitable for studying the contributions of non-translational isometries in PSH's by computer simulations since there is no problem in constructing hundreds of simulated comparable catalogues. A detailed account of the MPSH technique, including numeric simulation, can be found in [15] – [17].

Further research

Any other means of reducing the statistical noise may play the role of extracting all distance correlations instead of just those due to translations. Therefore, a good approach to this issue is perhaps to study quantitatively the noise of PSH's in order to develop filters. Another scheme for extracting these correlations from PSH's is to modify what we have defined to be an observed universe to make stronger the signal which results from the non-translational isometries. These ideas are currently under investigation by our research group.

The main disadvantages of the known statistical approaches to determine the topology of our universe from discrete sources are that they all assume that: (i) the scale factor

$a(t)$ is accurately known, so one can compute distances from redshifts; (ii) all objects are comoving to a very good approximation, so multiple images are where they ought to be; and (iii) the objects have very long lifetimes, so there exist images of the same object at very different distances from one of our images. These are rather unrealistic assumptions, and no method will be effective unless it abandons these simplifying premises. Our idea of a suitable choice of data in catalogues (defined to be a choice of observed universe) seems to be powerful enough to circumvent these problems. Indeed, if one takes as observed universe a thin spherical shell, instead of a ball, all the sources will be at almost the same distance from the observer, and (i) we do not need to know what this distance is since we can look for angular correlations between pairs of sources, instead of distance correlations, therefore avoiding the need of knowing the scale factor, (ii) it is unimportant whether they are comoving sources because all of them are now contemporaneous, and so (iii) it is also irrelevant if they have short lifetimes. In the thin spherical shell one can look for angular correlations among Γ -pairs instead of the distance correlations of the crystallographic method. This approach is also currently under study by our research group.

A Comparison

In what follows we shall discuss the connection between our results and those of references [5, 6].

The results we have derived regarding PSH's for hyperbolic manifolds do not match with the explanation given in [5] for the absence of spikes. Indeed, in [5] it is argued that two types of pairs of images can give rise to spikes, namely *type I* and *type II* pairs. Considering these types of pairs they argue that in their simulated PSH's built for the Weeks manifold there are no spikes because: (i) the number of *type I* pairs is too low; and (ii) *type II* pairs cannot appear in hyperbolic manifolds. It should be noticed from the outset that the *type II* pairs in [5] are nothing but the Γ -pairs of the present article, whereas *type I* pairs are not the uncorrelated pairs of this paper. Now, since we have shown that (i) Γ -pairs as well as uncorrelated pairs which arise (both) from the covering isometries give rise to the topological signature for multiply-connected manifolds, and (ii) the topological signal of g -pairs is a spike (in a PSH) only if g is a Clifford translation, then the only reason for the absence of spikes of topological origin in PSH's corresponding to hyperbolic manifolds is that there is no Clifford translation in hyperbolic geometry. Further, that the small number of *type I* pairs is not responsible for the absence of spikes is endorsed by the PSH for the Best manifold reported in [6], which was performed for an observed universe

large enough so that it contains in the mean approximately 30 topological images for each cosmic object; and yet no spikes whatsoever of topological origin were found in the PSH.

Using expression (4.13) for the EPSH one can also clarify the effect of subtracting from the PSH corresponding to a particular 3-manifold the PSH of the underlying simply connected covering manifold. This type of difference has been performed for simulated comparable catalogues (with equal number of images and identical cosmological parameters) for a Best and H^3 manifolds [6]. In general the plots of that difference exhibit a fraction $1/(N - 1)$ of the topological signature of the isometries plus (algebraically) the fluctuations corresponding to both PSH's involved, namely the PSH for the underlying simply connected space and the PSH for the multiply connected 3-manifold itself. To understand that this is so, let us rewrite eq. (4.2) as

$$\Phi(s_i) = \Phi_{exp}(s_i) + \rho(s_i) , \quad (6.1)$$

where $\rho(s_i)$ denotes the noise (statistical fluctuations) of $\Phi(s_i)$. Using the decomposition (6.1) together with (4.16) one easily obtains

$$\Phi(s_i) - \Phi^{sc}(s_i) = \frac{1}{N - 1} \varphi^S(s_i) + \rho(s_i) - \rho^{sc}(s_i) , \quad (6.2)$$

where $\varphi^S(s_i)$ denotes the topological signature, which is given by (4.16). According to (4.15), (4.16) and (6.1), had they examined the difference $\Phi(s_i) - \Phi_{exp}^{sc}(s_i)$, between the PSH built from that Best manifold and the EPSH for the corresponding covering space, they would have expurgated the noise $\rho^{sc}(s_i)$, and thus their plot (fig. 3 in [6]) would simply contain a superposition of the topological signature and just one noise, $\rho(s_i)$. As a matter of fact, the “wild oscillations in the scale of the bin width” they have found are caused by the superposition of the two statistical fluctuations $\rho^{sc}(s_i)$ and $\rho(s_i)$, whereas the “broad pattern on the scale of R_0 ” ought to carry basic features of the topological signature corresponding to the Best manifold they have examined. But again, a detailed account of these points is a matter that has been discussed in [15] – [17].

Finally, regarding the peaks of ref. [7] it is important to bear in mind that graphs tend to be effective mainly to improve the degree of intuition, to raise questions to be eventually explained, and for substantiating theoretical results. Often, however, they do not constitute a proof for a result, such as the EPSH (4.12) we have formally derived from rather general first principles. A good example which shows the limitation of the conclusions one can withdraw from such graphs comes exactly from the PSH shown in fig. 1 of ref. [7], where there is a significant peak which according to our results clearly *cannot* be of topological origin because it does not correspond to any translation. Note, however, that just by examining that graph one cannot at all decide whether that sharp peak is of topological

nature or arises from purely statistical fluctuations. In brief, just by examining PSH's one cannot at all distinguish between spikes of topological origin from sharp peaks of purely statistical nature — our statistical analysis of the distance correlations in PSH's elucidates the ultimate role of all types of isometries (in PSH's) and are necessary to separate the spikes (sharp peaks) of different nature.

Acknowledgements

We thank the scientific agencies CNPq, CAPES and FAPERJ for financial support. G.I.G. is grateful to Helio Fagundes for introducing him to the method of cosmic crystallography and to Evelise Gausmann for fruitful conversations.

Appendix A

Our aim in this appendix is to show that the $\eta_{exp}(s_i)$ can be decomposed into an uncorrelated part and a correlated part according to (4.3). To this end suppose that there are K comparable catalogues \mathcal{C}_k , each of which contains the same number N of sources and corresponds to the constant curvature 3-manifold manifold $M(t_0)$. Since any pair of sources in each catalogue is either a Γ -pair or an uncorrelated pair (a pair that is not a Γ -pair) the number $\eta^{(k)}(s_i)$ of pairs of sources in a catalogue \mathcal{C}_k with squared separations in J_i splits as

$$\eta^{(k)}(s_i) = \eta_u^{(k)}(s_i) + \frac{1}{2} \sum_{g \in \tilde{\Gamma}} \eta_g^{(k)}(s_i) , \quad (\text{A.3})$$

where $\eta_u^{(k)}(s_i)$ is the number of uncorrelated pairs of sources in \mathcal{C}_k with squared separations in J_i , and $\eta_g^{(k)}(s_i)$ is the number of g -pairs in \mathcal{C}_k whose squared separations are in J_i . $\tilde{\Gamma}$ is the covering group Γ without the identity map, and the factor $1/2$ in the sum accounts for the fact that, in considering all non-trivial covering isometries, we are counting each Γ -pair twice, since if (p, q) is a g -pair, then (q, p) is a (g^{-1}) -pair.

Taking the mean value of $\eta^{(k)}(s_i)$ in the set of K catalogues we have

$$\langle \eta(s_i) \rangle = \langle \eta_u(s_i) \rangle + \frac{1}{2} \sum_{g \in \tilde{\Gamma}} \langle \eta_g(s_i) \rangle , \quad (\text{A.4})$$

where

$$\langle \eta(s_i) \rangle = \frac{1}{K} \sum_{k=1}^K \eta^{(k)}(s_i) , \quad (\text{A.5})$$

and analogous expressions hold for $\langle \eta_u(s_i) \rangle$ and $\langle \eta_g(s_i) \rangle$. Now, since expected values the limit of their corresponding mean values when the number of samples K tends

to infinity, one has

$$\eta_{exp}(s_i) = \lim_{K \rightarrow \infty} \langle \eta(s_i) \rangle , \quad (\text{A.6})$$

and similar expressions hold for $\langle \eta_u(s_i) \rangle$ and $\langle \eta_g(s_i) \rangle$. Now, using (A.4) we obtain

$$\eta_{exp}(s_i) = \eta_u(s_i) + \frac{1}{2} \sum_{g \in \tilde{\Gamma}} \eta_g(s_i) , \quad (\text{A.7})$$

where $\eta_u(s_i)$ is the expected number of observed uncorrelated pairs of sources with squared separations in J_i , and $\eta_g(s_i)$ is the expected number of observed g -pairs whose squared separations are in J_i .

Appendix B

Our aim in this appendix is to show that equation (5.4) holds in first approximation. To this end let us write

$$N_k = N + \Delta N_k , \quad (\text{B.8})$$

for $k = 1, \dots, K$. Since we are assuming that ΔN_k is small ($\Delta N_k \ll N$), in first order approximation in ΔN_k the normalization coefficient of (5.2) can clearly be expanded to give

$$\frac{1}{N_k(N_k - 1)} \approx \frac{1}{N(N - 1)} - \frac{2N - 1}{[N(N - 1)]^2} \Delta N_k , \quad (\text{B.9})$$

so that the mean value of $\Phi(s_i)$ in this approximation reduces to

$$\langle \Phi(s_i) \rangle \approx \frac{2}{N(N - 1)} \frac{1}{\delta s} \left[\langle \eta(s_i) \rangle - \frac{2N - 1}{N(N - 1)} \langle \Delta N \eta(s_i) \rangle \right] , \quad (\text{B.10})$$

where

$$\langle \Delta N \eta(s_i) \rangle = \frac{1}{K} \sum_{k=1}^K \Delta N_k \eta^{(k)}(s_i) . \quad (\text{B.11})$$

However, since the quantities ΔN_k and $\eta^{(k)}(s_i)$ are statistically independent, then the equation

$$\langle \Delta N \eta(s_i) \rangle = \langle \Delta N \rangle \langle \eta(s_i) \rangle \quad (\text{B.12})$$

holds. Now, inserting this equation into (B.10) one obtains

$$\langle \Phi(s_i) \rangle \approx \left[1 - \frac{2N - 1}{N(N - 1)} \langle \Delta N \rangle \right] \frac{2}{N(N - 1)} \frac{1}{\delta s} \langle \eta(s_i) \rangle . \quad (\text{B.13})$$

In the limit $K \rightarrow \infty$ we have that $\langle \Delta N \rangle \rightarrow 0$ and $\langle \eta(s_i) \rangle \rightarrow \eta_{exp}(s_i)$. Thus, in first order approximation we have

$$\lim_{K \rightarrow \infty} \langle \Phi(s_i) \rangle \approx \Phi_{exp}(s_i) , \quad (\text{B.14})$$

which completes the proof.

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